Math in Moscow, 2014-15 academic year
Ordinary differential equations (http://math-info.hse.ru/s14/12)
Exercises for lesson 12: total recall (05/07/2015)
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Problem 1. Solve the following equations:

(a) $y' - y = 2x - 3$.
(b) $y' = (y - x)^{10} + 1$. Find a solution with initial condition $y(1) = 1$.
(c) $(x + 2y)dx - xdy = 0$;
(d) $2xydx + (x^2 - y^2)dy = 0$;

Problem 2. Find an equation of phase curves (no explicit form of phase curves are needed). Draw phase portrait.

(a) $\dot{x} = x^2, \quad \dot{y} = y(x + y)$;
(b) $\dot{x} = y^2 + 2y + 1, \quad \dot{y} = x^2 - 1$;
(c) $\dot{x} = 2y\cos^2 x, \quad \dot{y} = 1 + y^2 \sin 2x$;
(d) $\dot{x} = -y + 2x, \quad \dot{y} = x + 2y$;

Problem 3. Solve the following systems of ODE. Find the solution with initial condition

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0).$$

(a) $\dot{x} = 2x + y, \quad \dot{y} = 2y + z, \quad \dot{z} = 2z$;
(b) $\dot{x} = -x, \quad \dot{y} = 2x - y, \quad \dot{z} = 3x + y - z$. (Hint: instead of finding of Jordan basis one can decompose matrix of the system to the sum of scalar and nilpotent and use definition of matrix exponential directly.)

Problem 4. Investigate all singular points of the system. Detect their types and stability. For nodes and saddles, find eigenvectors of the linearization. Sketch phase portrait of the system near the singular points.

(a) $\begin{cases} \dot{x} = x^2 - y \\ \dot{y} = \ln(1 - x + x^2) - \ln 3 \end{cases}$
(b) $\begin{cases} \dot{x} = \ln(2 - y^2) \\ \dot{y} = e^x - e^y \end{cases}$
(c) $\begin{cases} \dot{x} = \ln \frac{y^2 - y + 1}{3} \\ \dot{y} = x^2 - y^2 \end{cases}$
(d) $\dot{x} = \ln(1 - y + y^2), \quad \dot{y} = 3 - \sqrt{x^2 + 8y}$.

Problem 5. Using Lyapunov theorem find for which values of the parameter s the singular point $(0, 0)$ is asymptotically stable? For which it is Lyapunov unstable? For which s the theorem will not give the answer?

$$\begin{cases} \dot{x} = -4e^x + 6\sin(y) + 4 \\ \dot{y} = -\sin(4y) + \sin(sx) \end{cases} \tag{1}$$