Math in Moscow, 2014-15 academic year
Ordinary differential equations (http://math-info.hse.ru/s14/12)
Assignment ODE-9 (To be returned 05/07/2015)
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Theorem 1 (Lyapunov). *Consider system $\dot{x} = v(x)$, $x(t) \in \mathbb{R}^n$, $v(0) = 0$, $\frac{\partial v}{\partial x}\big|_{x=0} = A$. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A. Then:*

- *if* $\operatorname{Re} \lambda_j < 0$ *for all j, then zero solution is asymptotically stable;*
- *if there exists j such that* $\operatorname{Re} \lambda_j > 0$*, then zero solution is Lyapunov unstable;*
- *otherwise Theorem gives no answer.*

1 (3 + 3 + 3). Use Lyapunov theorem to investiage the stability of zero solution depending on the values of the parameters a и b. For which a and b zero solution is asymptotically stable? Lyapunov unstable? For which values Lyapunov theorem gives no answer?

(a). $\begin{cases} \dot{x} = ax - 2y + x^2 \\ \dot{y} = x + y + xy \end{cases}$

(b). $\begin{cases} \dot{x} = y + \sin x \\ \dot{y} = ax + by \end{cases}$

(c). $\begin{cases} \dot{x} = ax + 2y \\ \dot{y} = -5x - 3y \end{cases}$

2 (4). Prove Lyapunov theorem for the case of linear systems with fixed coefficients and diagonalizable matrix (i.e. system of form $\dot{x} = Ax$, whre A is diagonalizable).

3 (3). Construct two one-dimensional differential equations $\dot{x} = v_1(x)$ and $\dot{x} = v_2(x)$ such that $v_j(0) = 0$ and $v_j'(0) = 0$ for $j = 1, 2$ and the origin is asymptotically stable for v_1 and Lyapunov unstable for v_2.

Remark 1. *This problem shows that the impossibility of detecting the stability type of the equilibrium knowing only linear part of the equation is essential: if you know only linear part of the equation at the equilibrium and it is not covered by Lyapunov theorem (i.e. there exists eigenvalue with zero real part), than you cannot decide, if the equilibrium stable or unstable. It is not just a drawback of Lyapunov theorem.*