Math in Moscow, 2014-15 academic year

Ordinary differential equations (http://math-info.hse.ru/s14/12)

Assignment ODE-8 (To be returned 04/23/2015)

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1 \(1 + 1 + 1\). Consider system

\[
\dot{x} = x - 3y, \quad \dot{y} = 3x + y, \tag{1}
\]

where \((x(t), y(t)) \in \mathbb{R}^2\).

(a) Find complex number \(\lambda = \alpha + i\omega\) such that system (1) is equivalent to equation

\[
\dot{z} = \lambda z, \tag{2}
\]

where \(z(t) = x(t) + iy(t), x(t)\) and \(y(t)\) are real.

(b) Find a solution of equation (2) with initial condition \(z(0) = z_0, z_0 \in \mathbb{C}\).

(c) Find a solution of equation (1) with initial condition \(x(0) = x_0, y(0) = y_0, (x_0, y_0) \in \mathbb{R}^2\).

2 \(1 + 2 + 1\). Assume that real \(2 \times 2\) matrix \(A\) has two distinct complex conjugated eigenvalues \(\lambda = \alpha + i\omega\) and \(\bar{\lambda} = \alpha - i\omega\). Let \(v\) be eigenvector corresponding to \(\lambda\).

(a) Prove that \(\bar{v}\) is eigenvector corresponding to \(\bar{\lambda}\).

(b) Prove that complex vector-functions \(w(t) = ve^{\lambda t}\) and \(\bar{w}(t) = \bar{v}e^{\bar{\lambda}t}\) are solutions of differential equation

\[
\dot{w} = Aw, \tag{3}
\]

where \(w(t) = (x(t), y(t)) \in \mathbb{C}^2\).

(c) Prove that real vector-functions \(\text{Re}w\) and \(\text{Im}w\) are also solutions of differential equations (3). (Hint: express \(\text{Re}w\) in terms of \(w\) and \(\bar{w}\)).

Remark 1. Any real solution of (3) can be expressed as linear combination of \(\text{Re}w\) and \(\text{Im}w\) with real coefficients.

3 (3 points each). Using previous problem and remark, find all real solutions of the following systems. Detect their types according to the classification (saddle/node/center/focus) and their stability in cases of node and focus.

(a) \(\dot{x} = -x - 2y, \quad \dot{y} = 4x + 3y\)

(b) \(\dot{x} = -x - 5y, \quad \dot{y} = x + y\)

(c) \(\dot{x} = 8x + 25y, \quad \dot{y} = -2x - 6y\)

(d) \(\dot{x} = 5x + 4y, \quad \dot{y} = -10x - 7y\)

4 (4). For which value of parameter \(\alpha\) the following system has singular point of type center?

\[
\dot{x} = ax + 2y, \quad \dot{y} = -5x - 3y
\]