Math in Moscow, 2014-15 academic year
Ordinary differential equations (http://math-info.hse.ru/s14/12)
Assignment ODE-8 (To be returned 04/23/2015)
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1 $(1+1+1)$. Consider system

$$\dot{x} = x - 3y, \quad \dot{y} = 3x + y, \tag{1}$$

where $(x(t), y(t)) \in \mathbb{R}^2$.

(a). Find complex number $\lambda = \alpha + i\omega$ such that system (1) is equivalent to equation

$$\dot{z} = \lambda z, \tag{2}$$

where $z(t) = x(t) + iy(t)$, $x(t)$ and $y(t)$ are real.

(b). Find a solution of equation (2) with initial condition $z(0) = z_0$, $z_0 \in \mathbb{C}$.

(c). Find a solution of equation (1) with initial condition $x(0) = x_0$, $y(0) = y_0$, $(x_0, y_0) \in \mathbb{R}^2$.

2 $(1+2+1)$. Assume that real 2×2 matrix A has two distinct complex conjugated eigenvalues $\lambda = \alpha + i\omega$ and $\bar{\lambda} = \alpha - i\omega$. Let v be eigenvector corresponding to λ.

(a). Prove that $\bar{v}$ is eigenvector corresponding to $\bar{\lambda}$.

(b). Prove that complex vector-functions $w(t) = ve^{\lambda t}$ and $\bar{w}(t) = \bar{v}e^{\bar{\lambda} t}$ are solutions of differential equation

$$\dot{w} = Aw, \tag{3}$$

where $w(t) = (x(t), y(t)) \in \mathbb{C}^2$.

(c). Prove that real vector-functions $\operatorname{Re} w$ and $\operatorname{Im} w$ are also solutions of differential equations (3). (Hint: express $\operatorname{Re} w$ in terms of w and $\bar{w}$.)

Remark 1. *Any real solution of* (3) *can be expressed as linear combination of* $\operatorname{Re} w$ *and* $\operatorname{Im} w$ *with real coefficients.*

3 (3 points each). Using previous problem and remark, find all real solutions of the following systems. Detect their types according to the classification (saddle/node/center/focus) and their stability in cases of node and focus.

(a). $\dot{x} = -x - 2y, \quad \dot{y} = 4x + 3y;$

(b). $\dot{x} = -x - 5y, \quad \dot{y} = x + y;$

(c). $\dot{x} = 8x + 25y, \quad \dot{y} = -2x - 6y;$

(d). $\dot{x} = 5x + 4y, \quad \dot{y} = -10x - 7y.$

4 (4). For which value of parameter α the following system has singular point of type *center*?

$$\dot{x} = \alpha x + 2y, \quad \dot{y} = -5x - 3y$$