Math in Moscow, 2014-15 academic year
Ordinary differential equations (http://math-info.hse.ru/s14/12)
Assignment ODE-4 (To be returned 12/03/2015)
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1 (6 total). Let $p = (x, y)$ and $v = (v_x, v_y)$. Which of the following functions

$$\omega\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

are 1-forms? Provide explanation for your answer. For every function which is 1-form express it in terms of dx and dy.

(a). $\omega(p, v) = x$;
(b). $\omega(p, v) = 2v_y$;
(c). $\omega(p, v) = 0$;
(d). $\omega(p, v) = xv_x^2 + yv_y^2$;
(e). $\omega(p, v) = x^2 v_x + y^2 v_y$;
(f). $\omega(p, v) = \frac{xv_x}{x^2+y^2+1}$;

Definition 1. Equation $y' = F(x, y)$ is called *homogeneous* if $F(\lambda x, \lambda y) = F(x, y)$ for every λ, x, y.

2 $(1 + 1 + 1 + 1 + 2 + 2 + 2)$. Consider equation

$$y' = \frac{y^2 - x^2}{2xy}. \tag{1}$$

(a). Plot several (at least three) isoclines for the direction field of equation (1). (You can find definition of isocline in assignment 3. It is stated there for vector fields but can be applied verbatim for direction fields.)
(b). Show that the equation is homogeneous.
(c). Sketch the direction field for this equation.
(d). Consider change of variables: $z = y/x$. Find the image of the isoclines found in (a) in coordinates (z, x).
(e). Put new unknown function $z = y/x$ into equation (1) (i.e. find the derivative of z assuming z is a function of x; note that y is also a function of x and assume that y' satisfies the equation (1). Then express the right-hand side in terms of z and x.)
(f). Solve this equation.
(g). Solve the initial equation (1).

Remark 1. *Any homogeneous equation can be solved by change of unknown function function $z = y/x$.*

3 $(2 + 2 + 2 + 3)$. Find integral curves.
(a). $(x^2 - 2xy)\,dx + (y^2 + 2xy)\,dy = 0$;
(b). $(2 - 9xy^2)x\,dx + (4y^2 - 6x^3)y\,dy = 0$;
(c). $\frac{y}{x}dx + (y^3 + \ln x)\,dy = 0$;
(d). $(-2xy^4 \sin(x^2 + y^4))\,dx + (4y^3(-y^4 \sin(x^2 + y^4) + \cos(x^2 + y^4)))\,dy = 0$;

Bonus part

4 (5). Consider 1-form $\omega = f(x, y)dx + g(x, y)dy$. Prove that if $f'_y = g'_x$ than there exists function $H(x, y)$ such that $dH = \omega$.