Math in Moscow, 2014-15 academic year
Ordinary differential equations
Assignment ODE-2 (To be returned 02/26/2015)

Ilya Schurov

1 (each part is 2 points). Find all solutions of an equation in form $y = y(x)$. Describe explicitly the domain of each solution. Plot some integral curves.

(a) $y' = y/(2x)$;  
(b) $y' = -y/x$;  
(c) $y' = -4x/y$;  
(d) $y' = -xy$.

2 (each part is 3 points). Solve the following equations (it is not needed to express the solution as a functions of $x$, the answer in implicit form (but without integrals) is acceptable):

(a) $xy' + y = y^2$;
(b) $(x + 2y)y' = 1$. Find a solution with initial condition $y(0) = -1$; (Hint: use appropriate substitution).
(c) $xy' = x + y$ (hint: consider substitution $z = y/x$);
(d) $y' = \sqrt{y}$. Find all solutions with initial condition $y(0) = 0$. (This problem is the same as 3c from assignment 1, you can skip it if you believe that you solved that problem correctly.)

3 (3+1+4). Denote by $x(t)$ the size of shop population at moment $t$, and assume that its normal growth rate is given by the formula $(1 - x)x$. Coefficient $(1 - x)$ here is due to the fact that resources (e.g. food) are limited: if the size of the population is small, then $1 - x \approx 1$ and we have almost Malthusian exponential growth, but as $x$ increases relative reproduction rate decreases. Assume that we additionally fish out some fixed amount of fish in every unit of time — denote this value by $c > 0$ (fishing quota). So the equation for $x$ is the following:

$$\dot{x} = x - x^2 - c$$

(a) For which pairs of value of the fishing quota and the initial population size the population survives forever?

(b) What is the maximal value $c_{\text{max}}$ of $c$ for which the population survives forever if the initial population is big enough?

(c) Sketch the integral curves of corresponding differential equation for different values of $c$. (No exact solution needed.) Consider three cases: $0 < c < c_{\text{max}}$, $c = c_{\text{max}}$ and $c > c_{\text{max}}$.

4 (4). The following Lemma was discussed on the lecture. Let $f$ be $C^1$-smooth function (i.e. $f'$ exists and is continuous) and $f(x_0) = 0$. Then there exists constant $C$ and interval $(x_1, x_2)$ such that $|f(x)| < C|x - x_0|$ for all $x \in [x_1, x_2]$.

On the lecture, we proved this Lemma using integration. Prove it using mean value theorem instead of integration.